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OPTIMUM STRUCTURAL DESIGN BASED ON RELIABILITY AND PROOF-LOAD TEST

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Abstract

An approach in structural optimization based on reliability analysis is presented with an emphasis on the use of the proof-load test. In particular, described in detail are methods of optimizing the structural weight subject to a constraint on the expected cost, which is an extended version of the constraint on the probability of failure. Depending on whether the structure is statically determinate or indeterminate, different methods of optimization are described.

Numerical examples indicate that the expense of performing the proof-load test is always well compensated by the improvement of structural reliability due to such a test. In fact, under the constraint of the same expected cost, significant weight savings can be expected of a structure with proof-load-tested components, compared with the optimum weight of the structure consisting of components that are not proof-load-tested. The extent to which such an extra weight saving can be achieved depends on a parameter pertaining to the importance of individual components relative to the cost of failure.

It is shown how the significant improvement of statistical confidence in the reliability estimate can be achieved on the basis of proof-load test. The question of how to deal with the statistical confidence of the load distribution is also discussed at length.

I. Introduction

Thermo-mechanical properties of materials used for the structure of a space vehicle, such as fracture strength, elastic modulus, deformation capacity, linear thermal coefficient of expansion, etc. (particularly those of composite materials), exhibit considerable statistical variations. Furthermore, aerospace environments as well as loading conditions involve a number of uncertainties; for example: temperatures generated by aerodynamic friction, dynamic pressures, axial accelerations, acoustic and vibration loads, etc.

This indicates that both strengths of a structure and loads acting on the structure should be treated as random variables and that the concept of structural reliability should be incorporated into the analysis of the structure and its (optimum) design. In fact, some work has been done in this direction[1-6] at different levels of sophistication of reliability analysis.

It should be observed, however, that major structural components of a space vehicle are usually tested individually or otherwise under simulated environmental and loading conditions before the vehicle is sent into the mission. Since such simulated tests or proof-load tests are indispensable parts of the current engineering task within a space program, it is extremely important that the effect of such tests be taken into account in the estimation of structural reliability and in the (optimum) structural design. The present study presents, for a given expected cost constraint, quantitative results of considerable weight saving and increased reliability by taking into consideration the proof-load test.

From the viewpoint of reliability analysis, the advantage of performing the proof-load test can be summarized as follows. The test can improve not only the reliability value itself but also the statistical confidence in such a reliability estimate. This is because the proof-load test eliminates structures with strength less than the proof load. In other words, the structure which passes the proof load test belongs to a subset, having the strength higher than the proof load, of the original population. Therefore, it is obvious that the reliability of a structure chosen from this subset is higher than that of a structure chosen from the original population. Furthermore, the proof-load test truncates the distribution function of strength at the proof load, hence alleviating the analytical difficulty of verifying the validity of a fitted distribution function at the lower tail portion where data are usually non-existent. Evidently the difficulty still remains in the selection of a distribution function for the load. However, the statistical confidence in the reliability estimation now depends mainly on the accuracy of the load prediction. The question of how to deal with the statistical confidence of the load distribution is discussed in Section VI, in which the Baysian approach is suggested.

The present paper: (1) develops an approach to an optimum design (either minimum weight design or minimum expected cost design) introducing the proof load as an additional design parameter, and (2) shows the practical advantage of the use of proof loads in terms of weight saving.

The importance of this proposed approach in structural design is emphasized, not only from the viewpoint of optimization, but also from the realization that it is the only rational approach in the face of various uncertainties and that it establishes a definite design procedure applicable to most aerospace structures.

Although the present study places its emphasis on the problem of optimization of aerospace structures, the principle involved can be applied to

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optimization problems in other engineering disciplines such as the optimum design of civil engineering structures, naval structures, ground vehicles, material handling equipment and electronic systems. In particular, the optimization can be highly significant, e.g., for electronic systems consisting of thousands of components when the cost of each component is so small in comparison with the total cost of the system that a relatively high level of proof load can be applied as will be shown later.

Civil engineering structures such as gantry towers at launching sites, buildings, bridges, etc., gain increasing significance for space technology application as, for instance, in relation to urban development, transportation systems and the like. Because of their characteristic construction processes, it is recognized that these structures usually undergo a tacit process of proof-load test during construction. If the structure does not fail during and upon completion of construction, it implies that all of its structural components and therefore the structure itself have sufficient strength to withstand at least the dead load. This is the information that must be taken into consideration as the lower bound of the strength distribution for the reliability estimate of an existing structure. In this respect, the authors believe that it is not entirely impossible and in fact advisable to devise an inexpensive method of more explicit proof-load test which can establish such a bound.

Furthermore, if a structure under construction survives a live load due to severe wind or earthquake acceleration, which are referred to as secondary live loads in many design codes but which are of primary importance for safety considerations of existing structures, the combined action of such a live load and of the dead load (existing at the time of occurrence of the live load) can be interpreted as a proof load test. The fact that the partially completed structure has survived such a proof load test should be taken into consideration in the reliability analysis since this fact usually makes it possible to establish a better lower bound of the strength of each of structural components (existing in the partially completed structure).

An important implication of this argument is that separate considerations are given to the safety of a structure during and after completion of its construction. This seems quite reasonable since the cost of detection possibly by means of the proof-load test and the cost of the replacement of that part of the structure which failed because of a member or members with insufficient strength may be absorbed as the construction cost, whereas any failure, after the structure is placed into service by the client, would produce much more serious contractual and socio-economic problems, possibly involving human lives.

II. Expected Cost of Failure and Optimum Design

To present the essence of the idea, consider the following form of expected cost EC of the vehicle, taking only the cost of failure and of proof load test into account, although more elaborate forms are obviously possible and may be desirable depending on the specific problem at hand,

$$EC = \sum_{i=1}^{n} \frac{p_{0i} C_{0i}}{1 - p_{0i}} + p_f C_F \tag{1}$$

where

n = number of major structural components constituting the vehicle

p_{0i} = probability of failure of a candidate for the i^{th} component under the stress S_{0i} due to proof load.

p_f = probability of failure of the (entire) structure.

C_F = cost of failure (loss of vehicle, prestige, etc.)

C_{0i} = cost of proof load tests including cost of loss of candidate components that failed to pass the test.

$p_{0i}/(1-p_{0i})$ = expected number of candidates for the i^{th} component that fail under S_{0i} before the one that can sustain S_{0i} is obtained.

These quantities and hence EC are functions of design parameters. Furthermore, the weight W of the structure is also a function of the same parameters. Therefore, the optimization of W (or EC) under a constraint on EC (or W) can be performed with respect to these parameters.

It is pointed out that the absolute value of C_F will have no effect in the following optimization process. Note that if the proof-load test is not performed, then $p_{0i} = 0$ in Eq. (1) and the formulation reduces to the minimum weight design under the constraint of probability of failure as discussed in Refs. 1-6.

Under the further simplifying assumptions, as used in Refs. 1-6, that the resisting strengths R_i (in terms of stress such as yield stress) of individual components ($i = 1, 2, \cdots, n$) are independent of each other as well as of the load S, the probability of failure p_f of the structure can be shown[7] to be

$$p_f = \int_0^\infty \left\{ 1 - \prod_{i=1}^{n} \left[1 - F_{R_i}(c_i x) \right] \right\} f_S(x)\, dx$$

$$\cong \sum_{i=1}^{n} \int_0^\infty F_{R_i}(x) f_{S_i}(x)\, dx \tag{2}$$

where

$F_{R_i}(\cdot)$ = distribution function of R_i

S = load applied to the structure

S_i = stress acting on the i^{th} component

$f_{S_i}(\cdot)$ = density function of S_i

c_i = constant associated with the i^{th} component

With the aid of the theoretical and experimental structural analysis, the stress S_i acting on the i^{th} component when the load S is applied to the structure is given by

$$S_i = \frac{c_i S}{g_i(A_i)} \tag{3}$$

in which A_i is a design parameter representing the size of the i^{th} component so that its weight W_i can be given as $W_i = b_i A_i$ [see Eq. (23)], c_i is a constant if the structure is statically determinate whereas it is a function of $A_1, \cdots, A_n$ if the structure is statically indeterminate, and $g_i(A_i)$ is a function of A_i the form of which depends on the nature of the i^{th} component; for example, $g_i(A_i) = A_i$ = cross-sectional area for truss-like structures.

Any method of structural analysis can be employed to obtain Eq. (3) including the finite element method which is used extensively in the stress analysis of aerospace structures.

The density function $f_{S_i}(x)$ of S_i can be obtained from the density function $f_S(x)$ of S through the transformation indicated in Eq. (3) and can be shown to be

$$f_{S_i}(x) = f_S\left[\frac{g_i(A_i)x}{c_i}\right]\left|\frac{g_i(A_i)}{c_i}\right| \tag{4}$$

As was discussed in detail in Ref. 7, the following points are to be noted in the derivation of Eq. (2): (1) the definition of structural failure is in accordance with the weakest link hypothesis, that is, the failure will take place if at least one of the components fails, (2) the assumption that R_i are independent each other is a conservative one, (3) the approximation indicated in Eq. (2) is also of conservative nature, (4) the load S can be interpreted as reference value of a system of proportional loading acting on the structure, and (5) if p_f in Eq. (2) is to represent the probability of failure of a structure subject to a sequence of N (statistical) loads, $f_S(x)$ should be replaced by the density function $f_{S^*}(x)$ of the maximum load S^* in such a sequence; for a sequence of N "independent" loads, each distributed as S, the density function $f_{S^*}(x)$ is

$$f_{S^*}(x) = \frac{d\left[F_S(x)\right]^N}{dx} = N\left[F_S(x)\right]^{N-1} f_S(x) \tag{5}$$

where $F_S(x)$ is the distribution function of S.

The previous studies[1-6] were all based on the weakest link hypothesis and on the assumption of R_i being independent of each other. The approximation indicated in Eq. (2) was also employed in these studies with the exception of the work by Moses and Kinser, in which, essentially, the exact integral expression [the second member of Eq. (2)] together with Eq. (5) was used in a different analytical approach. The weakest link hypothesis which is probably adequate to describe the failure condition for statically determinate structure is also adopted in the present study for analytical simplicity. It is pointed out, however, that some evidence exists as to its validity for statically indeterminate structures[8] as well.

Of considerable practical importance is the case where the structure is subjected to a number of mutually exclusive and independent proportional loading systems (e.g., unusually severe wind and extremely strong earthquake motion) with $S^{*(1)}, S^{*(2)}, \cdots, S^{*(k)}$ denoting the maximum reference values of these loading systems within a specified period of time. The probability of failure of the structure is then given by the sum $\sum_{j=1}^{k} p_f^{(j)}$, where $p_f^{(j)}$ is the probability of failure of the structure under $S^{*(j)}$ only.

Employing in Eq. (1) the approximation indicated in Eq. (2)

$$EC = \sum_{i=1}^{n} EC_i \tag{6}$$

with EC_i being the expected cost of the i^{th} component,

$$EC_i = \frac{p_{0i}C_{0i}}{1 - p_{0i}} + p_{fi}C_F$$

$$= (\gamma_i q_i + p_{fi})\, C_F \tag{7}$$

where

$$q_i = \frac{p_{0i}}{1 - p_{0i}} \tag{8}$$

with

$$\gamma_i = \frac{C_{0i}}{C_F} (\ll 1) \;;$$

this ratio indicates relative importance of the i^{th} component with respect to the cost of structural failure and

p_{fi} = probability of failure of the i^{th} component

$$= \int_0^\infty F_{R_i}(x) f_{S_i}(x)\, dx \tag{9}$$

The quantities EC and EC_i can be expressed in terms of the cost of failure C_F by dividing both sides of Eqs. (6) and (7) by C_F

$$EC^* = \sum_{i=1}^{n} EC_i^* \tag{10}$$

$$EC_i^* = \gamma_i q_i + p_{fi} \tag{11}$$

where

$$EC^* = \frac{EC}{C_F} \qquad \text{and} \qquad EC_i^* = \frac{EC_i}{C_F}$$

Equations (10) and (11) indicate an important conclusion that the absolute value of the cost of failure C_F has no effect on the optimization process. In the present optimization formulation, it is only necessary to know or to estimate the ratio γ_i of the component cost C_{0i} to the cost of failure C_F.

If the proof-load test is not performed, i.e., if $q_i = 0$, then EC_i^* represents the probability of failure of the i^{th} component and EC the probability of failure of the entire structure.

To perform the reliability-based optimum design, it is necessary to know the distribution function $F_{\underline{R}_i}(x)$ of the resisting strength $\underline{R}_i$ of the candidate for the i^{th} component (henceforth referred to as "the parent strength distribution of the i^{th} component"). It is assumed that, before the proof-load test, this distribution and therefore its mean value and standard deviation are known with sufficient statistical confidence on the basis of material tests, past experience, etc. The question as to what is considered to be a sufficient confidence is crucial and will be discussed in detail in Part 2 of Section VI.

Let e_i denote a design parameter indicating the stress level S_{0i} of proof load to be applied to the candidate for the component in terms of $\underline{R}_i^*$;

$$S_{0i} = e_i \underline{R}_i^* \tag{12}$$

and the central safety factor ν_i be defined as

$$\nu_i = \frac{\underline{R}_i^*}{S_i^*} \tag{13}$$

where $\underline{R}_i^*$ and S_i^* are the measures of location (such as the mean values $\overline{\underline{R}}_i$ and $\overline{S}_i$) of the distribution of $\underline{R}_i$ and S_i respectively (this notation for the mean value is used throughout the paper unless otherwise specified).

Once the factor of safety ν_i is specified, the candidate for the i^{th} component should be so designed that the measure of location S_i^* of the stress S_i acting on the i^{th} component is equal to $S_i^* = \underline{R}_i^*/\nu_i$. This is accomplished by choosing A_i that satisfies

$$g_i(A_i) = \frac{c_i S^*}{\underline{R}_i^*} \nu_i \tag{14}$$

Equation (14) is obtained from the following relationship by replacing S_i^* by $\underline{R}_i^*/\nu_i$,

$$S_i^* = \frac{c_i S^*}{g_i(A_i)} \tag{15}$$

which is in turn obtained from Eq. (3), where S_i^* and S^* should be the measures of location of the same kind such as the mean value.

It is interesting to note that upon substituting Eq. (14) into Eq. (4), the density function of S_i becomes

$$f_{S_i}(x) = f_S\left(\frac{S^* \nu_i}{\underline{R}_i^*} x\right) \left|\frac{S^* \nu_i}{\underline{R}_i^*}\right| \tag{16}$$

and it is free from c_i.

The probability of failure p_{0i} of a candidate for the i^{th} member is given by

$$p_{0i} = F_{\underline{R}_i}(S_{0i}) = F_{\underline{R}_i}\left(e_i \underline{R}_i^*\right) \tag{17}$$

and the probability element $f_{R_i}(x)\,dx$ of the resisting strengths R_i of the proof-load-tested i^{th} component is given by

$$\begin{aligned} f_{R_i}(x)\,dx &= P(x < \underline{R}_i \le x + dx \,|\, \underline{R}_i > S_{0i}) \\ &= \frac{P(x < \underline{R}_i \le x + dx,\ \underline{R}_i \ge S_{0i})}{P(\underline{R}_i \ge S_{0i})} \\ &= \frac{H(x - S_{0i}) f_{\underline{R}_i}(x)\,dx}{1 - F_{\underline{R}_i}(S_{0i})} \end{aligned} \tag{18}$$

from which it follows that

$$F_{R_i}(x) = \frac{H(x - S_{0i})\left[F_{\underline{R}_i}(x) - F_{\underline{R}_i}(S_{0i})\right]}{1 - F_{\underline{R}_i}(S_{0i})} \tag{19}$$

where

$P(E)$ = probability of event E

$P(E_1 | E_2)$ = conditional probability of E_1 given E_2

$P(E_1, E_2)$ = probability of simultaneous occurrence of E_1 and E_2

$H(x)$ = Heaviside unit step function.

Using Eq. (12) into Eqs. (18) and (19) one obtains

$$f_{R_i}(x) = \frac{H\left(x - e_i \underline{R}_i^*\right) f_{\underline{R}_i}(x)}{1 - F_{R_i}(e_i \underline{R}_i^*)} \tag{20}$$

$$F_{R_i}(x) = \frac{H\left(x - e_i \underline{R}_i^*\right)\left[F_{\underline{R}_i}(x) - F_{\underline{R}_i}\left(e_i \underline{R}_i^*\right)\right]}{1 - F_{\underline{R}_i}\left(e_i \underline{R}_i^*\right)} \tag{21}$$

The Heaviside unit step function in Eqs. (20) and (21) truncates the original strength distribution at the stress level of the proof load. It is for this reason that the distribution of R_i is referred to as the truncated strength distribution in the following.

Since it follows from Eqs. (8) and (17) that $q_i \equiv q_i(e_i)$ and from Eqs. (9), (16), and (21) that $p_{fi} \equiv p_{fi}(e_i, \nu_i)$, Eq. (11) can be written as

$$EC_i^* = \gamma_i q_i(e_i) + p_{fi}(e_i, \nu_i) \tag{22}$$

The optimization problem considered in the present study is either to minimize the structural weight subject to the constraint on the (relative) expected cost, or to minimize the (relative) expected cost subject to the constraint on the structural weight, both with respect to e_i and ν_i. Since the analytical technique employed here can be applied to either case, only the first is discussed. Furthermore, because the optimization analysis for statically determinate structures can be simplified considerably, discussions for statically determinate and indeterminate structures are given separately. It should be mentioned, however, that the technique for statically indeterminate structures is a general one that can also be applied to the optimum design of statically determinate structures. In fact, the optimum values in a later example for a statically determinate structure are all checked by the technique used for the statically indeterminate structure.

III. Optimum Design of Statically Determinate Structures

The optimization problem in this case can be stated as follows.

Minimize the weight of the form

$$W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} b_i A_i \tag{23}$$

subject to

$$EC^* = \sum_{i=1}^{n} EC_i^* \le EC_a^* \tag{24}$$

with

$$EC_i^* = \gamma_i q_i(e_i) + p_{fi}(e_i, \nu_i) \tag{25}$$

In Eq. (23), W_i is the weight of the i^{th} component and b_i is a known constant; for example, if a truss-like structure is considered, A_i is the cross-sectional area and $b_i = \ell_i \rho_i$ with ℓ_i and ρ_i being the (specified) length and density of the same component respectively.

Rewriting Eq. (14), one obtains

$$\nu_i = \frac{\underline{R}_i^* g_i(A_i)}{c_i S^*} \tag{26}$$

which indicates that ν_i is a function of A_i (and A_i only) since c_i is a constant for statically determinate structures. Hence, equivalent to Eq. (25),

$$EC_i^* = \gamma_i q_i(e_i) + p_{fi}(e_i, A_i) \tag{27}$$

where $p_{fi}(e_i, A_i)$ is used for $p_{fi}(e_i, \nu_i(A_i))$ for simplicity.

Now, the problem is to minimize W in Eq. (23) subject to the constraint Eq. (24), with EC_i^* given by either Eq. (25) or Eq. (27).

Since Eq. (23) is linear and there is only one constraint equation, Eq. (24) is always active; i.e., the equality sign of Eq. (24) holds at optimum.

The present problem can now easily be formulated with the aid of the variational principle. Although the method does not generate the solution explicitly, it indicates that, for a minimum weight design, the following relationships must be satisfied, under the assumption that EC_a^* is small compared with unity so that at optimum, the variation of W_i/W is small in comparison with that of EC_a^* (see Appendix).

$$\frac{\partial EC_i^*}{\partial e_i} = 0 \qquad (i = 1, 2, \cdots, n) \tag{28}$$

$$\frac{EC_i^*}{EC_a^*} = \frac{W_i}{W} \qquad (i = 1, 2, \cdots, n) \tag{29}$$

Equations (28) state that, for an optimum structural weight, the stress level of the proof load to be applied to individual components should also be optimum in the sense that the relative cost EC_i^* of individual components are minimized at that stress level. As an example, under the assumption that $\underline{R}_i$ and S_i are normally distributed with coefficients of variation of 20% for S_i and 5% for $\underline{R}_i$, the dependence EC_i^* on e_i using a specific value of ν_i (=1.6) is plotted in Fig. 1 with γ_i as a parameter. The locus of those points at which EC_i^* assumes minimum values (Curve I) plays an important role in the following optimization process. Those values of the proof load e_i that produce

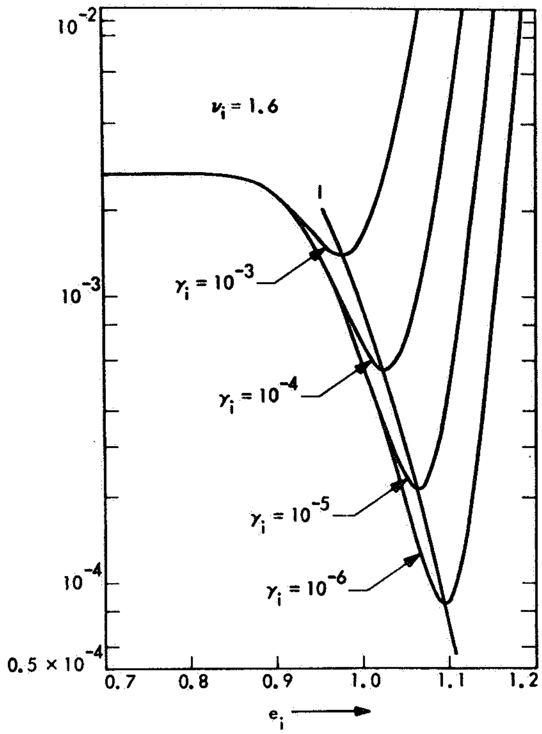

Fig. 1. Relative Expected Component Cost EC_i^* as a Function of Proof Stress Level e_i; Load Distribution is Normal with Coefficient of Variation 0.20 and Parent Strength Distribution is Normal with Coefficient of Variation 0.05

minimum EC_i^* associated with given ν_i and γ_i are denoted by e_i^*. This implies that Curve I indicates the relationship between EC_i^* and e_i^* given ν_i and γ_i.

Equations (29) state than an optimum weight is realized when the total weight is allocated to individual components in proportion to their expected costs. This fact was shown to be valid also for optimization without proof-load test[3] in which case the total weight was to be allocated proportionately to the probabilities of failure rather than to the expected costs.

Usefulness of Eqs. (28) and (29) lies in the fact that these can be used to develop an iterative procedure consisting of the following steps in arriving at a minimum weight design:

(1) Construct a diagram in which the $EC_i^* \sim e_i^*$ relationship is given for various values of ν_i and γ_i (Fig. 2). This is an extended version of Fig. 1.

(2) Try $EC_i^* = EC_a^*/n$ as a first estimate of EC_i^*.

(3) Read, from Fig. 2, e_i^* and ν_i corresponding to the latest estimate of EC_i^* and to the specified value of γ_i. A_i is then calculated from Eq. (14).

(4) Compute EC_i^* from Eqs. (29) using those values of A_i just obtained in (3).

(5) Go to (3) with the value of EC_i^* estimated in (4) and repeat the procedure.

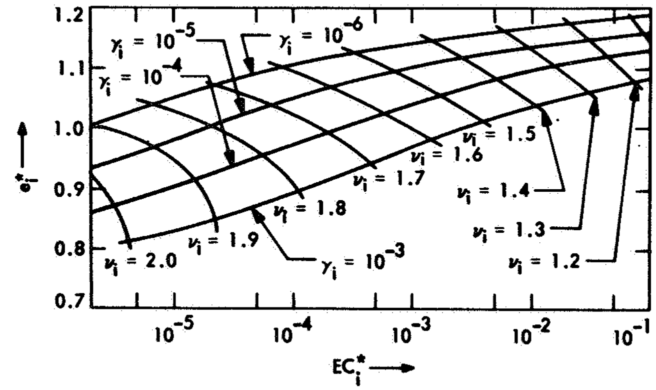

Fig. 2. Relative Expected Component Cost EC_i^* as a Function of Optimum Proof Stress Level e_i^*; Load Distribution is Normal with Coefficient of Variation 0.20 and Parent Strength Distribution is Normal with Coefficient of Variation 0.05

The rate in which this process converges to stable values of W_i and e_i is extremely rapid since the component weight W_i is insensitive to the variation of EC_i^*. In fact, experience shows that two cycles of iteration are sufficient to obtain the optimum design.

It is to be noted that in a numerical example given later, the information contained in Fig. 2 is stored in the computer memory and the third step of the above procedure is accomplished by means of an interpolation subroutine.

IV. Optimum Design of Statically Indeterminate Structures

For statically indeterminate structures, the central safety factors $\nu_i (i = 1, 2, \cdots, n)$ cannot be chosen arbitrarily since they should satisfy continuity equations. This situation can readily be demonstrated by a statically indeterminate truss subjected to a system of proportional loading with a reference value S. The applied stress S_i (and S_i^*) acting on individual members can be determined only after the cross-sectional area A_i of the members are specified. This implies that ν_i are functions of $A_1, A_2, \cdots, A_n$. In fact, c_i in Eq. (26) are, in general, functions of $A_1, A_2, \cdots, A_n$. Hence,

$$\nu_i = \nu_i(A_1, A_2, \cdots, A_n) \qquad i = 1, 2, \cdots, n \tag{30}$$

It is important to note that the inverse of Eq. (30) does not exist. In other words, one cannot express A_i as a function of $\nu_1, \nu_2, \cdots, \nu_n$ and therefore $\nu_i (i = 1, 2, \cdots, n)$ cannot be used as independent design variables as in the preceding section The optimization problem should therefore be stated as follows:

Minimize

$$W = \sum_{i=1}^{n} b_i A_i \tag{31}$$

subject to

$$EC^* = \sum_{i=1}^{n} EC_i^* \le EC_a^* \tag{32}$$

with

$$EC_i^* = \gamma_i q_i(e_i) + p_{fi}(e_i, \nu_i) \tag{33}$$

or

$$EC_i^* = \gamma_i q_i(e_i) + p_{fi}(e_i, A_1, A_2, \cdots, A_n) \tag{34}$$

The basic difference between the present case and that of statically determinate structures is quite clear; the probability of failure p_{fi} now depends not only on A_i and e_i but also on A_1, $A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n$. Unfortunately this fact makes it impossible to solve the present optimization ("design") problem in a simple iterative approach efficiently employed for statically determinate structures in the preceding section. It is pointed out in passing, however, that such a difference between statically determinate and indeterminate structures hardly makes it necessary to treat these structures differently in the reliability "analysis" where the essential problem is to estimate the probability of failure of "designed structures" as long as the weakest link hypothesis is assumed regardless of structural determinancy.

Nevertheless, the variational principle, as described in the Appendix for statically determinate structures, can be applied to the problem of indeterminate structures at least formally to obtain the following conditions for an optimum

$$\frac{\partial EC_i^*}{\partial e_i} = 0 \qquad (i = 1, 2, \cdots, n) \tag{35}$$

Hence, the problem is now to minimize W in Eq. (31) by a proper choice of A_i and e_i subject to a constraint on the expected cost Eqs. (34) and satisfying Eqs. (35).

It is believed that the optimization technique most appropriate to the present problem is a gradient move method as briefly described below.

In this method, a design point B_1 is first chosen arbitrarily in the acceptable domain defined by $EC^* < EC_a^*$ of the n-dimensional design space of $A_1, A_2, \cdots, A_n$.

Note that once B_1 is chosen, $A_1, A_2, \cdots, A_n$ are given and ν_i can be computed from Eq. (26). With these values of ν_i and the specified values of γ_i, EC_i^* and e_i^* can be read from Fig. 2. This makes it possible to check if B_1 is in the acceptable domain.

The design is then modified by moving normal to the weight contour by a specified step from point B_1 to a new design point B_2 with a lighter weight. This process is repeated until the constraint $EC^* = EC_a^*$ is reached at point B_0.

Let **U** and **V** be respectively the gradients of the relative expected cost EC^* [with e_i replaced by optimum values e_i^* satisfying Eq. (35)] and the weight W at point B_0 (and hence normal to the constraint $EC^* = EC_a^*$ and the weight contour);

$$\mathbf{U} = \nabla EC^* = \sum_{k=1}^{n} \frac{\partial EC^*}{\partial A_k} \mathbf{i}_k = \sum_{i}^{n}\sum_{k}^{n} \frac{\partial EC_i^*}{\partial \nu_i} \frac{\partial \nu_i}{\partial A_k} \mathbf{i}_k \tag{36}$$

$$\mathbf{V} = \nabla W = \sum_{k=1}^{n} \frac{\partial W}{\partial A_k} \mathbf{i}_k = \sum_{k=1}^{n} b_k \mathbf{i}_k \tag{37}$$

with $\mathbf{i}_k$ being the unit base vector in the positive direction of A_k axis.

Let **Q** be a vector such that

$$(\mathbf{U} \cdot \mathbf{Q})_{\text{at } B_0} \le 0 \tag{38}$$

$$(\mathbf{V} \cdot \mathbf{Q})_{\text{at } B_0} \le 0 \tag{39}$$

The direction of **Q** defines the so-called usable feasible direction.[9] A systematic scheme for finding **Q** has been proposed by Zoudendijk[9] and used in the present study.

Since EC_i^* are functions of e_i^* and ν_i, the partial derivatives

$$\left.\frac{\partial EC_i^*}{\partial \nu_i}\right|_{\text{at } B_0} \qquad (i = 1, 2, \cdots, n)$$

can be obtained from Fig. 2 by interpolation, whereas the partial derivatives

$$\left.\frac{\partial \nu_i}{\partial A_k}\right|_{\text{at } B_0} \qquad (i = 1, 2, \cdots, n, \; k = 1, 2, \cdots, n)$$

from Eq. (26) with the aid of the finite difference technique.

The design point is now moved from B_0 along **Q** in a specified step away from the constraint $EC = EC_a^*$ into the acceptable domain with a reduction in the weight. The modification of the design

proceeds along Q until the design point reaches the constraint again. Then, another usable feasible direction is found and the process is repeated until the design point B^* is reached on the constraint at which the Kuhn-Tucker optimality condition[9] is satisfied [Q cannot be found at B^* so as to satisfy Eqs. (38) and (39) simultaneously with at least one of them being purely an inequality]. This point B^* is an optimum design point corresponding to a local minimum for the weight W. The global minimum can usually be found as the least of the local minima obtained by beginning with a number of starting design points.

It should be mentioned that disregarding Eqs. (35), a straightforward application of the gradient move technique can be made by taking $e_i (i = 1, 2, \cdots, n)$ as independent design variables in the design space. Hence, a design point B_1 is first chosen arbitrarily in the acceptable domain defined by $EC^* < EC_a^*$ of the 2n-dimensional design space of $A_1, A_2, \cdots, A_n, e_i, e_2, \cdots, e_n$. Then, the procedure just described can be employed to obtain a local minimum. It is believed that the computational work involved in such an approach will be much more than the approach taking advantage of Eqs. (35). It is emphasized that Eqs. (35) provide not only the computational advantage but also a physical significance of the optimum test level of components, as discussed previously in Section III.

V. Numerical Examples

1. For the purpose of comparison, the same numerical example as in Ref. 6 is considered in which the minimum weight design is performed of a statically determinate, truss-like structure consisting of ten components and subjected to a system of proportional loading. Since the failure condition is assumed to be that of yielding, the resisting strength $R_i = \sigma_y$ = yield stress. The assumption of γ_i being a constant for all components is used for simplicity without loss of generality.

A constraint imposed on EC in this example is

$$EC \le 10^{-3} C_F \quad \text{or} \quad EC^* \le 10^{-3} \tag{40}$$

Note that, without the proof load test, this formulation reduces to the minimum weight design under the constraint of probability of failure $p_f \le 10^{-3}$ as discussed in Ref. 6.

The weight W can be written in the form of Eq. (23) with n = 10 and

$$W_i = \rho L_i A_i = \frac{\rho L_i c_i \overline{S} \nu_i}{\overline{\sigma}_y} \tag{41}$$

where

ρ = density of material = 0.283 lb/in.^{3}

L_i = length of the i^{th} component = 60 in.

$\overline{S}$ = mean value of S = 60×10^3 psi

$\overline{\sigma}_y$ = mean value of yield stress $\sigma_y = 40 \times 10^3$ psi

$c_i = 0.1 \times i (i = 1, 2, \cdots, 10)$

It is assumed that the distribution functions of S and σ_y are both normal with the coefficient of variation 0.20 and 0.05 respectively. Then, a choice of a set of values for ν_i and S_{0i} determines the distribution function of R_i (that is, the truncated distribution of σ_y). This in turn makes it possible to evaluate p_{fi} [Eq. (9)]. Therefore, in this example, the independent design parameters are ν_i and e_i or $S_{0i} = e_i \overline{R}_i = e_i \overline{\sigma}_y$.

The minimum weight design subject to the constraint of Eq. (24) can now easily be achieved by the iterative method described in Section III.

The result is shown in Table 1 where the constraint of $p_f \le 10^{-3}$ is used for both conventional design and standard optimum design (without proof load test). Since all these designs are associated with the expected cost of $10^{-3} C_F$, Table 1 indicates the fact that, by performing the proof-load test, not only the reliability of the structure increases, but also considerable weight saving is achieved.

It is further observed from Table 1 that the extent of weight saving and reliability increase depend essentially on the value of γ_i, which is the ratio of the cost of the i^{th} component with respect to the cost of failure. For smaller values of γ_i, larger proof loads can be applied with the same constraint of the expected cost, thus yielding lighter structural weight (this can easily be realized from Fig. 2) and smaller probability of failure. For example, the structural weight W and the probability of failure p_f associated with the optimum structural design without proof load test are 253.2 lb and 10^{-3} respectively, while those associated with the optimum structural design with proof load test are 221 lb and 0.613×10^{-3} for $\gamma_i = 10^{-6}$, and 243.9 lb and 0.625×10^{-3} for $\gamma_i = 10^{-4}$.

Therefore, as a result of the proof-load test, one concludes that, for the optimum design, higher benefit can be obtained for smaller values of γ_i. It is due to this conclusion that the optimum design with proof load test can be highly significant for electronic systems consisting of thousands of components when the cost of each component is very small compared to the total cost of the system.

The study in Ref. 6 showed that more weight saving can be expected if the more accurate expression rather than the approximation is used for evaluating the probability of failure [Eq. (2)], although then Eqs. (29) will no longer be valid and hence a more elaborate computational scheme has to be applied.

2. A (statically indeterminate) three-member truss is designed for a minimum weight to resist a set of proportional loading as shown in Fig. 3. The mean value of yield stress for each member is 40×10^3 psi and that of S is 100×10^3 psi. The constraint on the cost EC is $5 \times 10^{-4} C_F$. Both σ_y and S are normal with coefficients of variation 0.20 and 0.05 respectively.

The weight function is

$$\frac{W}{\rho L} = (2)^{1/2} A_1 + A_2 + (2)^{1/2} A_3 \tag{42}$$

where W = total weight, ρ = density of material, L = length of member 2 and A_i = area of the i^{th} member.

Table 1. Ten-Member Structure ($EC_a^* = 10^{-3}$)

Member i	Mean load, $\bar{S}$	Conventional design (equal safety factor)	Standard optimum design	Current optimum design (with proof-load test)							
				$\gamma_i = 10^{-6}$		$\gamma_i = 10^{-5}$		$\gamma_i = 10^{-4}$		$\gamma_i = 10^{-3}$	
		A_i, in.^{2}	A_i, in.^{2}	A_i, in.^{2}	ϵ_i^*	A_i, in.^{2}	ϵ_i^*	A_i, in.^{2}	ϵ_i^*	A_i, in.^{2}	ϵ_i
1	0.1	0.274	0.287	0.257	1.064	0.271	0.998	0.283	0.920	0.287	0.84
2	0.2	0.547	0.562	0.498	1.079	0.524	1.019	0.550	0.941	0.561	0.86
3	0.3	0.817	0.833	0.734	1.079	0.771	1.031	0.812	0.954	0.831	0.87
4	0.4	1.09	1.101	0.966	1.087	1.013	1.039	1.068	0.963	1.097	0.88
5	0.5	1.37	1.367	1.196	1.096	1.252	1.044	1.322	0.971	1.361	0.88
6	0.6	1.64	1.630	1.424	1.100	1.490	1.049	1.573	0.977	1.622	0.89
7	0.7	1.92	1.893	1.65	1.103	1.725	1.053	1.821	0.982	1.882	0.89
8	0.8	2.19	2.153	1.875	1.105	1.959	1.056	2.068	0.986	2.140	0.90
9	0.9	2.46	2.413	2.098	1.107	2.192	1.058	2.313	0.990	2.397	0.90
10	1.0	2.74	2.672	2.320	1.109	2.423	1.061	2.556	0.993	2.653	0.91
	Structural weight (W) lb	255.6	253.2	221.0		231.3		243.9		251.8	
	Probability of failure (p_f)	1.0×10^{-3}	1.0×10^{-3}	0.613×10^{-3}		0.6158×10^{-3}		0.625×10^{-3}		0.839×10^{-3}	

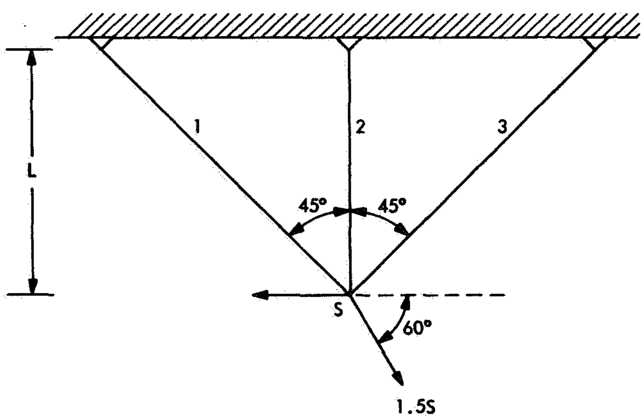

Fig. 3. Three-Member Truss

The gradient move method described in the preceding section is employed to find an optimum design. It is found that, under the loading condition described in Fig. 3, the minimum weight design is the one for which the area of member 1 is zero. Or, in this particular case, the minimum weight design is a statically determinate structure. For the purpose of comparison, the result is listed in Table 2 for different values of γ_i (γ_i are assumed equal for i = 1, 2, 3) as well as for zero proof load (standard optimum design). Again, considerable amount of weight saving is accomplished for smaller values of γ_i.

Although the possibility of buckling is not considered here, it can be treated without any difficulty.[7]

3. A spherical shell of constant thickness fixed around its edge and subjected to a uniformly distributed load S is to be designed for a minimum weight (Fig. 4). The mean values of the yield stress σ_y and the applied load S are respectively $\bar{\sigma}_y = 45 \times 10^3$ psi (for both tension and compression) and $\bar{S} = 0.6 \times 10^3$ psi. The constraint on the expected cost EC is $EC_a = 10^{-5}\ C_F$. Both σ_y and S are normally distributed with coefficient of variation 0.05 and 0.2 respectively. The maximum stress σ_{max} due to load S is the meridional stress at the fixed edge and approximately equal to[10]

$$\sigma_{max} = -Sf(a, \alpha, h) \tag{43a}$$

with

$$f(a, \alpha, h) = \left(\frac{a \sin \alpha}{h}\right)^2 \left[0.75 - 0.038 \left(\frac{a \sin \alpha}{h}\right)^2 \sin^2 \alpha\right]$$

for

$$\frac{a \sin^2 \alpha}{h} < 3 \tag{43b}$$

Table 2. Three-Member Truss ($EC_a^* = 0.5 \times 10^{-3}$)

Member	Standard Optimum Design	$\gamma_i = 10^{-6}$		$\gamma_i = 10^{-5}$		$\gamma_i = 10^{-4}$		$\gamma_i = 10^{-3}$	
	A_i, in.^{2}	A_i, in.^{2}	e_i^*	A_i, in.^{2}	e_i^*	A_i, in.^{2}	e_i^*	A_i, in.^{2}	e_i^*
1	0	0	0	0	0	0	0	0	0
2	4.568	3.94	1.119	4.102	1.075	4.320	1.011	4.515	0.93
3	1.579	1.37	1.107	1.433	1.059	1.512	0.991	1.568	0.907
$\frac{W}{\rho L}$	6.802	5.88		6.129		6.459		6.732	
P_f	5×10^{-4}	3×10^{-4}		2.8×10^{-4}		3.2×10^{-4}		3.8×10^{-4}	

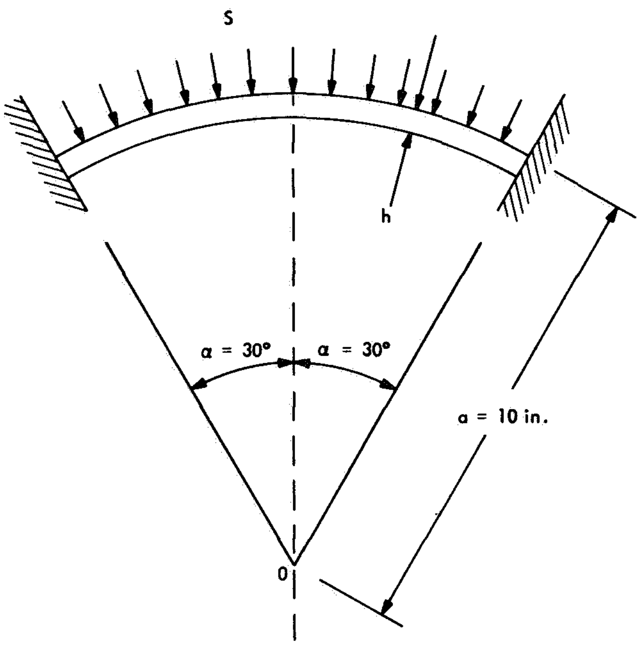

Fig. 4. Spherical Shell of Constant Thickness

and

$$f(a, \alpha, h) = 1.2 \frac{a}{h}$$

for

$$3 \le \frac{a \sin^2 \alpha}{h} < 12 \qquad (43c)$$

where a is the shell radius.

With the aid of Eqs. (43), the mean applied stress $\bar{\sigma}_{max}$ can be defined as the maximum stress produced by the mean applied load $\bar{S}$; $\bar{\sigma}_{max} = \bar{S}\, f(a, \alpha, h)$. The central safety factor ν is then $\nu = \bar{\sigma}_y / \bar{\sigma}_{max}$ and the stress level S_0 due to proof load is $S_0 = e\bar{\sigma}_y$.

Since this is a one-component structure, the optimum design can be achieved without using the gradient move method. The procedure is as follows:

1. Construct a diagram where the $EC_i^* \sim e_i^* \sim \nu_i$ relationship is given for various values of γ_i (Fig. 2).

2. Read e^* and ν from Fig. 2 for specified constraint EC_a^* and given value of γ.

3. With the safety factor ν just evaluated, the thickness h of the shell is computed using the following expression obtained from Eqs. (43).

$$f(a, \alpha, h) = \frac{\bar{\sigma}_y}{\nu \bar{S}} \qquad (44)$$

The results are listed in Table 3 for various values of γ including the case of standard optimum design ($e^* = 0$) for the purpose of comparison.

4. A spherical shell with variable thickness and subjected to a uniformly distributed load S is designed for a minimum weight (Fig. 5). The thickness at the top of the shell is h_1 while it is h_2 at the clamped edge. The thickness varies linearly with respect to the angle α. No attempt is made here for the determination of the optimum shape of the shell (for this aspect, the reader is referred to Ref. 11). The following values are used for numerical computation; $\bar{S} = 380$ psi, $\bar{\sigma}_y = 40 \times 10^3$ psi, $E = 30 \times 10^6$ psi, $G = 12 \times 10^6$ psi, $\rho = 0.238$ lb/in.^{3} Both σ_y and S are assumed to be normally distributed with coefficient of variation 0.2 and 0.05 respectively.

Table 3. Spherical Shell with Constant Thickness
($EC_a^* = 1.0 \times 10^{-5}$)

	Standard optimum design	$\gamma = 10^{-6}$	$\gamma = 10^{-5}$	$\gamma = 10^{-4}$	$\gamma = 10^{-3}$
h, in.	0.315	0.28	0.297	0.31	0.314
e^*	0	1.05	0.983	0.895	0.824
ν	1.971	1.75	1.854	1.941	1.964
p_f	1×10^{-5}	5.9×10^{-6}	5.6×10^{-6}	6.7×10^{-6}	7.2×10^{-6}

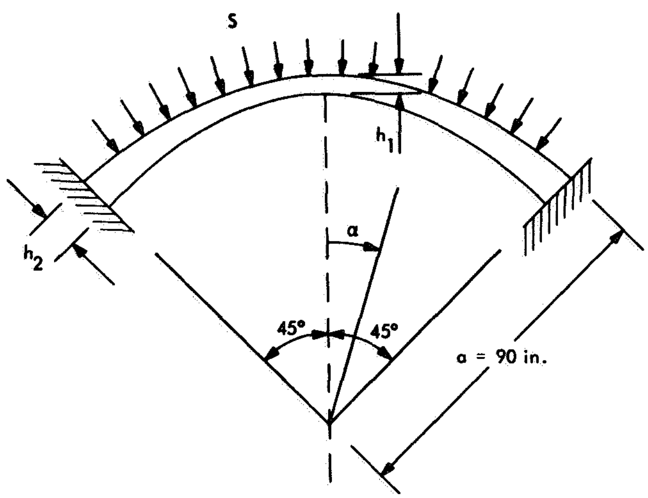

Fig. 5. Spherical Shell of Variable Thickness

The weight function is

$$W = 2\pi a^2 \rho \left\{ \left[1 - \frac{4}{\pi(2)^{1/2}} \right] h_1 + \left[\frac{4}{\pi(2)^{1/2}} - \frac{1}{(2)^{1/2}} \right] h_2 \right\} \tag{45}$$

where a is the shell radius. The finite element method is employed for stress analysis of the shell while the gradient move method is used to find the optimum design. The maximum stress occurs either at the clamped edge (meridional stress) or at the top of the shell (tangential stress) depending on the magnitude of h_1 and h_2.

It is noted that a number of starting design points were tried all resulting in the same optimum design. The results are listed in Table 4 for different values of γ_i and EC_a^*. For an easy reference, optimum values of thickness (h_1 and h_2) and the weight (W) are plotted as functions of EC_a^* in Figs. 6 and 7, respectively.

VI. Discussion

1. The result of the preceding examples indicates, as expected, that the level of proof load to be applied to individual components is lower for more important components with a larger value of γ_i and higher for less important components with a smaller value of γ_i, reflecting a simple fact that the more expensive the component is, the less one can afford a possibility to lose it by the proof load test.

For instance, in the truss considered in Example 2, under the same constraint of $EC_a^* = 5 \times 10^{-4}$, the optimum levels of proof load to be applied to member 3 are $e^* = 0.907$ for $\gamma = 10^{-3}$ and $e^* = 1.107$ for $\gamma = 10^{-6}$. Similarly, the optimum values of e_i are 0.824 for $\gamma = 10^{-3}$ and 1.05 for $\gamma = 10^{-6}$ in the spherical shell of Example 3.

2. The statement is made in the Introduction that the proof-load test can improve the statistical confidence in the reliability estimate because the test truncates the distribution function of the strength, hence alleviating, if not completely removing, the difficulty of justifying the use of a fitted distribution function at the lower tail portion where data are usually non-existent.

The validity of this statement evidently rests on whether the truncated strength distribution can really be established with a significantly improved confidence on a sample of practical size. In general, this can be achieved if the magnitude of the proof load is reasonable in the sense that it is equal to a strength value within a central portion of the parent strength distribution, since then the significant part of the truncated distribution in connection with the evaluation of probability of failure (strength values larger than but close to the point of truncation or the proof load) involves neither extreme lower nor extreme upper tail of the parent strength distribution. Note that this is equivalent to a statement that p_{0i} should not be too small compared with or too close to unity. For example, when considering a coefficient of variation of parent strength distribution of 0.05, values of e_i^* in the $\pm 2\sigma$ range (between $1.0 - 2 \times 0.05 = 0.90$ and $1.0 + 2 \times 0.05 = 1.10$) may be regarded as reasonable. Hence, the optimum levels of proof load test obtained for member 3 in Example 2 are reasonable. However, the optimum level for $\gamma = 10^{-3}$ in Example 3 is not reasonable because the proof load is so small that a sample of unreasonably large size would still be required to establish the strength distribution although truncated by the proof load. Similar situation exists when the level of proof load is unreasonably high.

Table 4. Optimum Design (h_1, h_2 in in., W in lb)

$\gamma = 10^{-6}$				
EC_a^*	0.3×10^{-4}	0.3×10^{-3}	0.3×10^{-2}	0.3×10^{-1}
h_1	0.755	0.698	0.626	0.547
h_2	1.694	1.532	1.372	1.187
ν	1.678	1.511	1.349	1.171
e^*	1.073	1.112	1.149	1.174
W	5798	5266	4716	4078
p_f	1.706×10^{-5}	2.22×10^{-4}	1.62×10^{-3}	2.64×10^{-2}
$\gamma = 10^{-5}$				
EC_a^*	0.3×10^{-4}	0.3×10^{-3}	0.3×10^{-2}	0.3×10^{-1}
h_1	0.804	0.7136	0.647	0.561
h_2	1.791	1.596	1.415	1.221
ν	1.768	1.573	1.392	1.203
e^*	1.011	1.072	1.111	1.145
W	6139	5465	4867	4205
p_f	1.59×10^{-5}	1.76×10^{-4}	2.26×10^{-3}	2.46×10^{-2}
$\gamma = 10^{-4}$				
EC_a^*	0.3×10^{-4}	0.3×10^{-3}	0.3×10^{-2}	0.3×10^{-1}
h_1	0.85	0.752	0.665	0.581
h_2	1.88	1.672	1.474	1.264
ν	1.855	1.657	1.45	1.242
e^*	0.932	1.009	1.068	1.108
W	6451	5732	5058	4351
p_f	1.99×10^{-5}	1.67×10^{-4}	1.96×10^{-3}	2.38×10^{-2}
$\gamma = 10^{-3}$				
EC_a^*	0.3×10^{-4}	0.3×10^{-3}	0.3×10^{-2}	0.3×10^{-1}
h_1	0.859	0.796	0.700	0.603
h_2	1.912	1.753	1.548	1.316
ν	1.885	1.730	1.527	1.296
e^*	0.857	0.927	1.003	1.061
W	6554	6021	5314	4529
p_f	2.76×10^{-5}	2.23×10^{-5}	1.90×10^{-3}	2.19×10^{-2}
Standard Optimum Design				
$EC_a^* = p_f$	0.3×10^{-4}	0.3×10^{-3}	0.3×10^{-2}	0.3×10^{-1}
h_1	0.860	0.80	0.728	0.650
h_2	1.916	1.771	1.616	1.421
ν	1.888	1.749	1.592	1.399
e^*	0	0	0	0
W	6566	6078	5541	4886

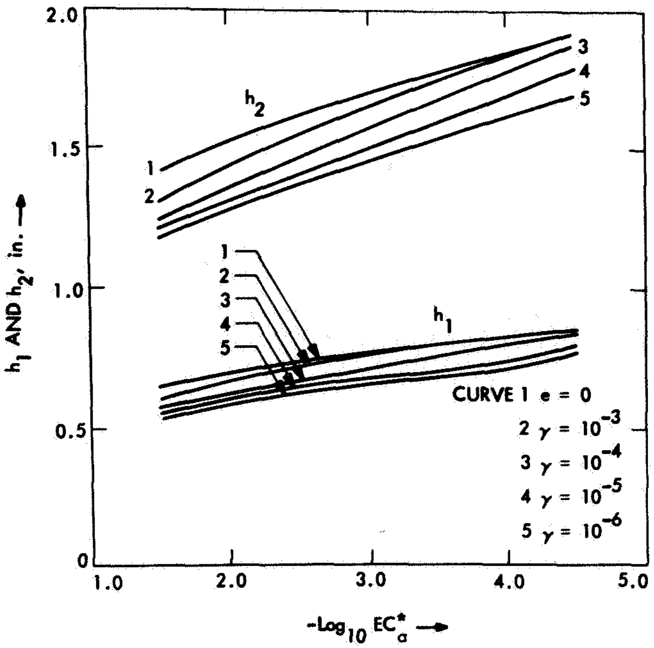

Fig. 6. Optimum Thickness h_1 and h_2 of Spherical Shell of Variable Thickness vs Expected Cost Constraint EC_a^*

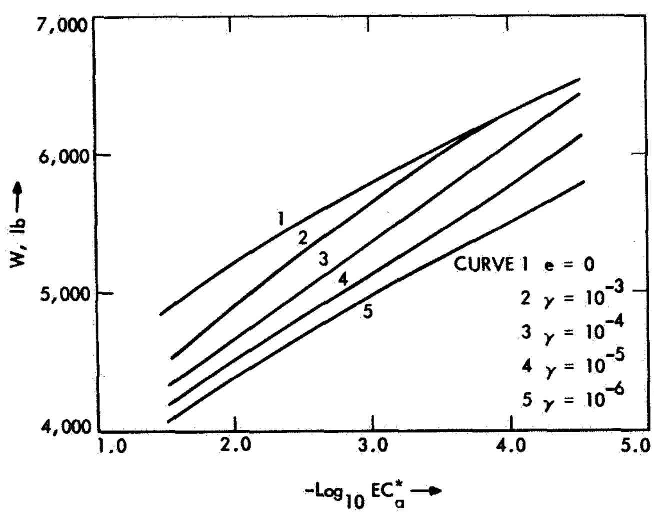

Fig. 7. Optimum Weight W of Spherical Shell of Variable Thickness vs Expected Cost Constraint EC_a^*

This observation makes it necessary to emphasize the tacit assumption employed in the preceding examples that the cost C_{Ei} of the test for establishing the truncated strength distribution of the i^{th} component (this test should not be confused with the proof-load test) is independent of the point of truncation (e_i or S_{0i}) yet to be determined. Such an assumption, employed in the present study for first approximation, is believed to be valid when the statistical properties of the strength of the material used is well known on the accumulated data or when the optimum levels of the proof load are found to be within the $\pm 2\sigma$ range of the parent strength distribution as a result of the analysis. Evidently, the cost $C_{Ei}(e_i)$ of the test for establishing the truncated strength distribution with a reasonable confidence is expected to increase rapidly as the optimum location of the truncation increases or decreases beyond the $\pm 2\sigma$ range, particularly when the material to be used is new to the engineers.

It is strongly emphasized that the assumption of C_{Ei} being independent of e_i is not essential in the present formulation and analysis. In fact, the effect of the cost C_{Ei} can easily be included in the formulation by modifying Eq. (1) into the form

$$EC = \sum_{i=1}^{n} \left[\frac{p_{0i} C_{0i}}{1 - p_{0i}} + C_{Ei}(e_i) \right] + p_f C_F \qquad (46)$$

provided, of course, that the functional form of $C_{Ei}(e_i)$ is reasonably well known.

The formulation based on Eq. (46) is much more likely to produce optimum values for e_i within a central portion of the parent strength distribution because other values of e_i incur extremely large cost in $C_{Ei}(e_i)$ and therefore are unlikely to become optimum values.

This fact is of utmost importance. Indeed, it is because of this that the optimum design based on reliability and proof-load test is expected to be insensitive to the tail portions of the parent strength distribution.

Even in the formulation without involving the cost C_{Ei}, most of the cases demonstrated in Examples 1-4 produced reasonable optimal levels of proof load implying that most of these optimum solutions were insensitive to the analytical form of the parent strength distribution.

This is the main reason why the normal distribution was assumed for the parent strength distribution without apparent justification at the outset. Another reason is that all of the previous work also assumed the normal distribution for the parent strength distribution and therefore the same had to be assumed in the present study for possible comparison.

It is recommended, however, that the Pearson distribution family[12] be used if the first four moments of the distribution are known reasonably well.

3. In the present approach, the strength within an individual member is assumed to be invariant (though statistical) for simplicity. Such an assumption is, however, subject to a critical observation that in reality the strength usually varies statistically from point to point even within the same member. It should be recognized, therefore, that the failure of the member actually occurs once the resisting strength at any location within the member is exceeded by the load (in terms of stress, strain, or displacement depending on the definition of failure) acting at that point. For example, when the load is uniformly distributed, as in a truss member, the resisting strength R_i of the i^{th} member can be defined as the strength of the weakest cross-section of the member and, therefore, may be distributed according to one of the asymptotic

distribution functions of the smallest values. This implies a possible necessity of taking the statistical size effect into consideration for a more rigorous analysis.

It is pointed out that a similar but possibly more complicated situation arises if, for example, elastic moduli of the material are more realistically treated as statistical functions of spacial coordinates. In such cases, the coefficients of the constitutive equations become statistical functions of spacial coordinates, hence making the stress analysis untractable, at least at present time, even under the assumption that the statistical functions are homogeneous.

4. The normal distribution is used for the load as an example simply because, as for the strength distribution, the same was used in the previous work. The use of the normal distribution is, however, not essential for the development of the present analysis. Therefore, other distribution functions should be used if there is any reason to believe empirically or theoretically that they represent the statistical load better than the normal distribution.

The following discussion proceeds under the assumption that the load is normally distributed. However, the discussion applies in principle to the case where the load is distributed otherwise.

It is only in rare occasions that even a reasonable amount of data exist for the environmental condition of a specific space mission. For the prediction of the load therefore, engineers usually must depend on incomplete knowledge and past experience if such has been accumulated on similar missions. Under these circumstances, the reliability of the two parameters (the mean value μ_S and the standard deviation σ_S) of the normal distribution assumed for the load suffers from a considerable lack of statistical confidence.

One possible way to cope with this kind of situation seems to be a sensible use of the Baysian approach in which these parameters are treated as if they were random variables with a joint density function $\phi(x, y)$ (x for μ_S and y for σ_S) constructed so as to reflect both the past experience and the accuracy of the load prediction. If a set of observed data directly related to the load are somehow available, such information should be used to modify the density function $\phi(x, y)$, following the Bayes theorem, by multiplying it by the likelihood of the observed data.

It is now clear that the minimum weight W^* computed in the preceding section is a conditional one under a given set of μ_S and σ_S; $W^* \equiv W^*(\mu_S,\sigma_S)$. In other words, depending on the values of the parameters μ_S and σ_S, the minimum weight assumes different values. It is important to realize that $W^*(\mu_S,\sigma_S)$ can be interpreted as "the best design if the mean and the standard deviation of the load distribution are truly μ_S and σ_S."

The question immediately arises then which design one should choose in the face of uncertainty involved in the mean value and the standard deviation. A possible answer to this question is as follows. First, construct a loss function that represents, in some general sense, the loss L caused by the choice of a specific minimum weight design W_0^* in place of $W^*(\mu_S,\sigma_S)$ that should be chosen if the mean and the standard deviation were known to be μ_S and σ_S respectively.

The analytical form of the loss function would probably depend on the managerial as well as engineering judgment except for the fact that it is a function of $|W_0^* - W^*(\mu_S,\sigma_S)|$.

Once the form of the loss function is constructed, then "a best design W_0^* in the face of uncertainty on the mean and the standard deviation" is chosen as the design that minimizes the expected loss EL or the expected value of L with respect to μ_S and σ_S. In other words, W_0^* satisfies

$$\frac{\partial EL}{\partial W_0^*} = 0 \qquad (47)$$

where

$$EL = E\left\{L\left[|W_0^* - W^*(\mu_S, \sigma_S)|\right]\right\}$$

$$= \iint_D \phi(x, y) L\left[|W_0^* - W^*(x, y)|\right] dx\, dy \qquad (48)$$

with D being the two dimensional domain in which x and y are defined.

For example, if a simple quadratic loss is assumed for L as a first approximation,

$$L = \left[W_0^* - W^*(\mu_S, \sigma_S)\right]^2 \qquad (49)$$

without discriminating an under-weight design against an over-weight design, then it follows immediately that

$$W_0^* = E\left[W^*(\mu_S, \sigma_S)\right] \qquad (50)$$

To compute the expected value in Eq. (50), the minimum weight $W^*(\mu_S,\sigma_S)$ has to be evaluated at a reasonable number of sets of values of μ_S and σ_S. This implies that the optimization procedures described in the preceding sections must be repeated the same number of times. The cost of performing such computation, however, may or may not be justified. For practical purposes, $W^*(\bar{\mu}_S,\bar{\sigma}_S)$ may be a good approximation for $E[W^*(\mu_S,\sigma_S)]$ although it is not quite clear at this time how accurate this approximation is.

In this context, the result of the preceding numerical examples can be considered as $W^*(\bar{\mu}_S,\bar{\sigma}_S)$ if the numerical values used for the mean and the standard deviation of the load distribution are interpreted as the best estimate $\bar{\mu}_S$ (denoted by $\bar{S}$ in the preceding sections) and $\bar{\sigma}_S$ of μ_S and σ_S respectively.

It is pointed out, that such an approximation is in general reasonable in view of the fact that the

analytical form of the loss function itself is a product of subjectively inclinded engineering, economical and managerial judgment.

An alternative approach of choosing design is that of the Baysian confidence as described below.

Consider, for example, a set of parameter values $\mu_{S\beta}$ and $\sigma_{S\beta}$ such that

$$P(\mu_S < \mu_{S\beta}, \sigma_S < \sigma_{S\beta}) = \beta \tag{51}$$

Furthermore, consider a design with the minimum weight $W^*(\mu_{S\beta}, \sigma_{S\beta})$ assuming that $\mu_{S\beta}$ and $\sigma_{S\beta}$ are the mean value and the standard deviation of the load distribution. Then, the probability that the best minimum weight $W^*(\mu_S, \sigma_S)$ associated with (unknown) true parameter values μ_S, σ_S will be less than $W^*(\mu_{S\beta}, \sigma_{S\beta})$, is β. It is assumed in the above statement that the smaller the mean value and the standard deviation are, the smaller the resulting minimum weight is. A design associated with $\mu_{S\beta}$ and $\sigma_{S\beta}$ can be considered reasonable if the confidence coefficient is small. It is to be noted that in this approach, the design becomes more conservative as a smaller value of β is specified.

5. At the last stage of this investigation, a paper by Barnett and Hermann[13] came to the attention of the present authors. It is acknowledged that the paper recognizes the practical significance and importance of proof testing from the viewpoint of reliability analysis and suggests a method of component optimization on the basis of the Weibull strength distribution [see part 3 of this section in this respect].

The present paper, however, explicitly describes methods of reliability-based optimum design of structures (consisting of a number of components) subjected to a "statistical load" and also explicitly formulates the problem of optimization in a framework of general cost-effectiveness approach. Furthermore, the present discussion on the statistical confidence of strength as well as load distribution [parts 2 and 4 of this section] is more complete.

6. Obviously, the present analysis is valid only for those cases where the quasi-static structural analysis can reasonably well replace the dynamic analysis, as exemplified by the structural response analysis of a spacecraft to the dynamic pressure which builds up as a function of time in such a way that it will produce no significant dynamic effect.

Also, in the present paper, it is assumed that the structural analysis accurately describes the stress and/or strain within the structure. The consideration for the error in the structural analysis is beyond the scope of this paper.

VII. Conclusion

An approach in structural optimization based on reliability analysis is presented with an emphasis on the use of proof-load test. In particular, methods of optimizing the structural weight subject to a constraint on the expected cost is described in detail. Different methods of optimization, depending on whether the structure is statically determinate or indeterminate, are described. The formulation of optimization problems using the constraint on the expected cost as defined in the present study is more general than using the constraint on the probability of failure as employed in the existing literature in the sense that the former reduces to the latter if no proof-load test is performed.

Numerical examples, with a particular but reasonable expression for the expected cost, indicate that the expense of performing the proof-load test is always well compensated by the improvement of structural reliability due to such a test. In fact, under the constraint of the same expected cost, significant weight savings can be expected of a structure with proof-load-tested components, compared with the optimum weight of the structure consisting of components that are not proof-load-tested. The extent to which such an extra weight saving can be achieved depends on a parameter pertaining to the importance of individual components relative to the cost of failure.

Also, as long as optimum levels of proof loads turn out to be within a central portion of the strength distribution, the proof-load test improves confidence of the estimated reliability value of the structure, since if this happens, the confidence in the reliability estimation depends mainly on the accuracy of the load prediction. An approach, in which such a restriction on the optimum level of proof load for the analysis to be meaningful can automatically be eliminated, is described.

The question of how to deal with the statistical confidence of the load distribution is also discussed at length.

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Appendix

Since at optimum $\delta W = 0$ whereas $\delta EC^* = 0$ by constraint,

$$\sum_{i=1}^{n} b_i \delta A_i = 0 \tag{A-1}$$

$$\sum_{i=1}^{n} \frac{\partial EC_i^*}{\partial e_i} \delta e_i + \sum_{i=1}^{n} \frac{\partial EC_i^*}{\partial A_i} \delta A_i = 0 \tag{A-2}$$

Hence,

$$\sum_{i=1}^{n} \left[\lambda \frac{\partial EC_i^*}{\partial e_i} \delta e_i + \left(b_i + \lambda \frac{\partial EC_i^*}{\partial A_i} \right) \delta A_i \right] = 0$$

from which it follows that

$$\frac{\partial EC_i^*}{\partial e_i} = 0 \qquad \text{for } i = 1, 2, \cdots, n \qquad \text{(A-3)}$$

$$b_i + \lambda \frac{\partial EC_i^*}{\partial A_i} = 0 \qquad \text{for } i = 1, 2, \cdots, n \qquad \text{(A-4)}$$

where λ is the Lagrange multiplier.

Equations (A-4) reduce to

$$\frac{\partial EC_1^*}{\partial W_1} = \frac{\partial EC_2^*}{\partial W_2} = \cdots = \frac{\partial EC^*}{\partial W} \qquad \text{(A-5)}$$

and hence

$$\frac{\Delta EC_i^*}{\Delta EC_a^*} = \frac{\Delta W_i}{\Delta W} \qquad i = 1, 2, \cdots, n \qquad \text{(A-6)}$$

Furthermore, it is observed from Fig. 2 [the relationship of EC_i^*, e_i^* and ν_i for a given value of γ_i in Eq. (1), for which Eqs. (A-3) are satisfied] that a small change in ν_i (which is proportional to W_i) results in a significant change in EC_i^* and similarly a small change of EC_i^* results in a negligible change in ν_i. Therefore, it is reasonable to assume that the variation of $W_i/\Sigma W_i$ due to that of EC_a^* is approximately zero (the smaller the value of EC_a^*, the better this approximation will be), i.e.,

$$\delta \left(\frac{W_i}{W} \right) \cong 0 \qquad \text{(A-7)}$$

$$\therefore \frac{W_i + \Delta W_i}{W + \Delta W} = \frac{W_i}{W}$$

Hence

$$\frac{\Delta W_i}{\Delta W} = \frac{W_i}{W} \qquad \text{(A-8)}$$

A sufficient condition for Eqs. (A-4) [or Eqs. (A-6)] to be satisfied, given Eq. (A-7) [or Eq. (A-8)], is

$$\frac{EC_i^*}{EC_a^*} = \frac{W_i}{W} \qquad \text{(A-9)}$$

This statement can be verified as follows. If Eq. (A-9) is valid,

$$\delta \left(\frac{EC_i^*}{EC_a^*} \right) = \delta \left(\frac{W_i}{W} \right) \qquad \text{(A-10)}$$

and, hence, according to Eq. (A-7)

$$\delta \left(\frac{EC_i^*}{EC_a^*} \right) = 0$$

Therefore,

$$\frac{\Delta EC_i^*}{\Delta EC_a^*} = \frac{EC_i^*}{EC_a^*} \qquad \text{(A-11)}$$

Equations (A-6) are automatically satisfied because of Eqs. (A-8), (A-9), and (A-11). Q.E.D.

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